XII CONFERENCE

Applications of Logic in Philosophy and in the Foundations of Mathematics

Szklarska Poręba 7–11 May 2007

MODAL HYBRID LOGICS

Andrzej Indrzejczak

Department of Logic
University of Łódź
Kopcińskiego 16/18
90–232 Łódź
Poland
e-mail: indrzej@filozof.uni.lodz.pl

HYBRID LOGIC IN A NUTSHELL

1. The main problem:

Asymmetry between local perspective of relational semantics and global perspective of standard modal language.

2. Bad results:

- many semantic features have no adequate representation
- problems with suitable modal proof theory

3. Remedy:

Explicit syntactic representation of states of a model is needed to get enough flexibility.

4. Realisation:

- external: e.g. Gabbay's LDS's
- internal = HYBRID LOGICS!

5. Advantages:

- more expressive language
- better behaviour in completeness theory – more straightforward and in fact complete theory of frame definability due to improved expressive power of the language
- proof theory – more natural and simpler
- complexity often untouched – basic logic still decidable (sat-problem PSPACE-complete as in standard modal logics)

WHAT'S NEXT

1. Brief history

2. Basic hybrid language and its semantics

3. Basic hybrid logic

4. Basic Extensions

   - Extra modalities

   - Binders

   - 1-order hybrid logic

5. Decidability, complexity, interpolation

6. Proof methods

- Labels
- Sequent Calculi
- Natural Deduction
- Tableau Systems
- Resolution
- Hybrid RND-system
- Extensions

7. Case study: Branching Time

8. Case study: Interval Tense Logic

HISTORY

A. Prior (58, 67) – the concept of four grades of tense-logical involvement (68)

J. McCarthy, P. Hayes (69) – situation calculus

R. Bull (70) – "history variables" for paths in branching tense logic

N. Rescher, A. Urquhart (71) – Topological Logic

M. Fitting (72) – prefixed modal tableau systems

J. Allen (84) – "Holds" operator

Sofia school (Gargov, Tinchev, Passy, Goranko) (85, 87, 90, 94) – works on CPDL (Combinatory PDL), theory of binders

J. Perzanowski (89) – general theory of modal operators ("makings") in ontology

D. Gabbay (96) – general theory of labelled deductive systems

Genuine Hybrid Logic Movement (Blackburn, Seligman, Areces) (since 92)

Arthur Prior's concepts

McTaggart's analysis of time versus Prior's logical systems:

- A-series (past, present, future) – T-calculus (tenses)

- B-series (earlier/later) – I-calculus (instants) (later called U-calculus)

I-calculus is more expressive than T-calculus

Prior's problem: how to show the primacy of T-calculus over I-calculus?

The solution is to develop I-calculus inside T-calculus via extension of the language – the third grade of tense-logical involvement (instant-variables and $\forall$).

BASIC MODAL AND TENSE LOGIC

Let $\mathbf{L_M}$ denote standard modal propositional language i.e. abstract algebra of formulae

$$\langle FOR, \neg, \wedge, \vee, \rightarrow, \Box, \Diamond \rangle$$

with denumerable set of propositional variables.

$$VAR = \{\ p, q, r, \ldots, p_1, q_1, \ldots\} \subseteq FOR$$

Nonatomic formulae are defined in ordinary way:

- if $\varphi \in FOR$, then $\odot\varphi \in FOR$, where $\odot \in \{\neg, \Box, \Diamond\}$

- if $\varphi \in FOR$ and $\psi \in FOR$, then $(\varphi \odot \psi) \in FOR$, where $\odot \in \{\wedge, \vee, \rightarrow\}$

$\mathbf{L_T}$ is the bimodal variant of $\mathbf{L_M}$ with Priorean operators G, F, H, P

Relational Semantics

The Modal Frame $\mathfrak{F} = \langle\ \mathcal{W}, \mathcal{R}\rangle$, where $\mathcal{W} \neq \varnothing$ is the set of states (worlds), and $\mathcal{R}$ is a binary relation on $\mathcal{W}$, called *accessibility relation*.

The Temporal Frame $\mathfrak{T} = \langle\ \mathcal{T}, <\rangle$, where $\mathcal{T} \neq \varnothing$ is the set of time-instants and $<$ is a binary relation on $\mathcal{T}$ − *the flow of time relation*.

A model on the frame $\mathfrak{F}$ (or $\mathfrak{T}$) is any structure $\mathfrak{M} = \langle\ \mathfrak{F}, V\rangle$, where V is valuation function on atoms ($V : AT \longrightarrow \mathcal{P}(\mathcal{W})$). For Tense logic models are defined analogously on $\mathfrak{T}$.

Model Satisfiability

Satisfaction of formulae in states of a model is defined as follows:

$\mathfrak{M}, w \models \varphi$	iff	$w \in V(\varphi)$ for any $\varphi \in AT$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \vee \psi$	iff	$\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \rightarrow \psi$	iff	$\mathfrak{M}, w \not\models \varphi$ or $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \Box\varphi$	iff	$\mathfrak{M}, w' \models \varphi$ for any w' such that $\mathcal{R}(ww')$
$\mathfrak{M}, w \models \Diamond\varphi$	iff	$\mathfrak{M}, w' \models \varphi$ for some w' such that $\mathcal{R}(ww')$

and for temporal operators:

$\mathfrak{M}, t \models G\varphi$	iff	$\mathfrak{M}, t' \models \varphi$ for any t' such that $t < t'$
$\mathfrak{M}, t \models F\varphi$	iff	$\mathfrak{M}, t' \models \varphi$ for some t' such that $t < t'$
$\mathfrak{M}, t \models H\varphi$	iff	$\mathfrak{M}, t' \models \varphi$ for any t' such that $t' < t$
$\mathfrak{M}, t \models P\varphi$	iff	$\mathfrak{M}, t' \models \varphi$ for some t' such that $t' < t$

Validity

We have also the concept of global (model) satisfiability and of (frame or class of frames) validity:

$\mathfrak{M} \models \varphi$ iff $\forall_{w \in \mathcal{W}_{\mathfrak{M}}}, \mathfrak{M}, w \models \varphi$ (or $\|\varphi\|_{\mathfrak{M}} = \mathcal{W}$)

$\mathfrak{F} \models \varphi$ iff $\forall_{\mathfrak{M} \in MOD(\mathfrak{F})}, \mathfrak{M} \models \varphi$

where $MOD(\mathfrak{F})$ is the set of $\mathfrak{M}$ built on $\mathfrak{F}$

$\mathcal{F} \models \varphi$ iff $\forall_{\mathfrak{F} \in \mathcal{F}}, \mathfrak{F} \models \varphi$

$\models \varphi$ iff $\forall_{\mathfrak{M}}, \mathfrak{M} \models \varphi$

$\mathbf{K}$ denote the set of all valid formulae in $\mathbf{L_M}$ and $\mathbf{K}\mathrm{t}$ denote the set of all valid formulae in $\mathbf{L_T}$

$\models_S \varphi$ iff $\forall_{\mathfrak{M} \in S}, \mathfrak{M} \models \varphi$

where S is suitable class of structures (models based on a class of frames)

Entailment

1. φ <u>follows locally</u> from Γ:

$\Gamma \models \varphi$ iff $\forall_{\mathfrak{M} \in S}(\|\Gamma\|_{\mathfrak{M}} \subseteq \|\varphi\|_{\mathfrak{M}})$
(or $\forall_{\mathfrak{M} \in S}, \forall_{w \in \mathcal{W}_{\mathfrak{M}}}$ (if $\mathfrak{M}, w \vDash \Gamma$, then $\mathfrak{M}, w \vDash \varphi$))

where S is suitable class of structures (models based on a class of frames)

2. φ <u>follows globally</u> from Γ:

$\Gamma \mid\models \varphi$ iff $Mod(\Gamma) \subseteq Mod(\varphi)$
(or $\forall_{\mathfrak{M} \in S}$ (if $\mathfrak{M} \vDash \Gamma$, then $\mathfrak{M} \vDash \varphi$))

where $Mod(\varphi) = \{\mathfrak{M} : \mathfrak{M} \vDash \varphi\}$

Note! if $\Gamma \models \varphi$, then $\Gamma \mid\models \varphi$

Limits of expressive strength of ordinary modal and tense language

φ defines the class of structures $\mathcal{F}$ iff
$\forall_{\mathfrak{F}\in\mathcal{F}}, \mathfrak{F} \models \varphi$

Some undefinable 1-order conditions

name	condition
irreflexivity	$\forall x \neg \mathcal{R}xx$
asymmetry	$\forall xy(\mathcal{R}xy \to \neg \mathcal{R}yx)$
antisymmetry	$\forall xy(\mathcal{R}xy \wedge x \neq y \to \neg \mathcal{R}yx)$
intransitivity	$\forall xyz(\mathcal{R}xy \wedge \mathcal{R}yz \to \neg \mathcal{R}xz)$
r. directedness	$\forall xy \exists z(\mathcal{R}xy \wedge \mathcal{R}zy)$
dichotomy	$\forall xy(\mathcal{R}xy \vee \mathcal{R}yx)$
trichotomy	$\forall xy(\mathcal{R}xy \vee \mathcal{R}yx \vee y = z)$
r. discreteness	$\forall xy(\mathcal{R}xy \to$ $\exists z(\mathcal{R}xz \wedge \neg \exists v(\mathcal{R}xv \wedge \mathcal{R}vz)))$

BASIC HYBRID LOGIC

We get basic hybrid propositional modal language $\mathbf{L_H}$ by adding to $\mathbf{L_M}$ (or $\mathbf{L_T}$):

a) the second sort of propositional symbols called *nominals*. We assume denumerable set $NOM = \{i, j, k, ...\}$ such that $VAR \cap NOM = \varnothing$; $VAR \cup NOM = AT$ is the set of atomic formulae. Members of NOM are introduced for naming states of a model domain

b) denumerable collection of unary satisfaction operators indexed by nominals $i:$ (or $@_i$). The new clause for nonatomic formulae is:

- if $\varphi \in FOR$ and $i \in NOM$, then $i : \varphi \in FOR$

and it reads "formula φ is satisfied in state i".

Every formula built with nominals and constants only is called *pure formula*, every formula of the shape $i : \varphi$ or $\neg i : \varphi$ is called *sat-formula*.

Some examples:

$\Diamond(i \wedge p)$ – neither pure nor sat-formula

$i \to \Diamond j$ – pure but not sat-formula

$i : (p \to \Diamond q)$ – sat- but not pure formula

$i : j$, $i : \Diamond j$ – both pure and sat-formulae with important functions (express identity and accessibility of states respectively)

Note some important features of $\mathbf{L_H}$:

- both nominals and satisfaction operators are genuine language elements not an extra metalinguistic machinery

- although nominals are terms they are treated as ordinary sentences

Remark: some authors (Tzakova, Demri) prefer to have weaker language with only nominals added but without satisfaction operators as a basic hybrid language. In what follows we use $\mathbf{L_{H^*}}$ to denote such a language.

Semantics

The concept of a frame is the same as in ordinary normal modal (or tense) logics. A model on the frame $\mathfrak{F}$ is any structure $\mathfrak{M} = \langle\ \mathfrak{F}, V\rangle$, where V is valuation function on atoms ($V : AT \longrightarrow \mathcal{P}(\mathcal{W})$) such that for any $i \in NOM$, $V(i)$ is a singleton.

Satisfaction of new formulae in states of a model is defined as follows:

$$\begin{array}{lll} \mathfrak{M}, w \vDash i & \text{iff} & \{w\} = V(i) \text{ for any } i \in NOM \\ \mathfrak{M}, w \vDash i : \varphi & \text{iff} & \mathfrak{M}, w' \vDash \varphi \text{ where } \{w'\} = V(i) \end{array}$$

The concept of global (model) satisfiability and of (frame) validity is the same.

Note that:

$\mathfrak{M}, w \vDash i : \varphi$ iff $\mathfrak{M} \vDash i : \varphi$

Some important features:

1. Internalization of local discourse – nominals give direct representation of states in a language (storing model data)

2. Possible jumping to already specified states in a model (retrieving model data)

3. Internalization of $\models$ by $i : \varphi$

4. Representation of identity theory (for states) by $i : j$

5. Internalization of accessibility relation by $i : \Diamond j$

One drawback: Tree model property fails!

Points 2-5 due to presence of satisfaction operators.

BASIC HYBRID LOGIC

Note that satisfaction operators are indeed modal – in fact normal modal – constants; they satisfy:

(K) $i : (\varphi \to \psi) \to (i : \varphi \to i : \psi)$ and

(RG) $i : \varphi$ is valid whenever φ is valid

Let $\mathbf{K_H}$ denote the set of all valid formulae in $\mathbf{L_H}$.

Note: $\mathbf{K} \subseteq \mathbf{K_H}$

$\mathbf{K_H}$ decidable (PSPACE-complete as ordinary $\mathbf{K}$)

Expressivity

1. New tautologies – an example:

$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$

2. New frame-defining formulae – e.g.:

Irreflexivity – $i \rightarrow \neg\Diamond i$

Asymmetry – $i \rightarrow \neg\Diamond\Diamond i$

Complete Hilbert Calculus for $\mathbf{K_H}$

$\mathbf{HK_H}$ – axiomatic version of $\mathbf{K_H}$ consists of:

1. Axioms of $\mathbf{CPL}$

2. Axioms of $\mathbf{K}$:

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

(Pos) $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$

3. Specific Hybrid Axioms:

(K:)	$i : (\varphi \to \psi) \to (i : \varphi \to i : \psi)$
(Selfdual:)	$i : \varphi \leftrightarrow \neg i : \neg\varphi$
(Intro:)	$i \wedge \varphi \to i : \varphi$
(Ref:)	$i : i$
(Agree)	$i : j : \varphi \leftrightarrow j : \varphi$
(Back)	$\Diamond i : \varphi \to i : \varphi$

4. Rules:

(MP) $\vdash \varphi \rightarrow \psi, \vdash \varphi \ / \ \vdash \psi$

(RG) $\vdash \varphi \ / \ \vdash \Box\varphi$

(RG:) $\vdash \varphi \ / \ \vdash i : \varphi$

(SUB) $\vdash \varphi \ / \ \vdash e(\varphi)$, where $e : VAR \longrightarrow FOR$, but $e : NOM \longrightarrow NOM$

Theorem (Completeness): The above axiomatic system is strongly complete for $\mathbf{K_H}$

but something more is needed for extensions of $\mathbf{K_H}$, let $\mathbf{HK_H^+}$ be $\mathbf{HK_H}$ with 2 additional rules:

(NAME) $\vdash i : \varphi \ / \ \vdash \varphi$, provided $i \notin \varphi$

(BG) $\vdash i : \Diamond j \rightarrow j : \varphi \ / \ \vdash i : \Box\varphi$, provided $i \neq j$ and $j \notin \varphi$

Lemma: Th($\mathbf{HK_H}$)=Th($\mathbf{HK_H^+}$)

Lemma: The following are $\mathbf{HK_H}$-theses:

(Sym:) $i : j \leftrightarrow j : i$

(Tran:) $i : j \wedge j : k \rightarrow i : k$

(Nom1) $j : \varphi \wedge j : i \rightarrow i : \varphi$

(Nom2) $j : \varphi \wedge i : j \rightarrow i : \varphi$

(Bridge) $\Diamond i \wedge i : \varphi \rightarrow \Diamond \varphi$

Lemma: The following rules are admissible in $\mathbf{HK_H}$ (derivable in $\mathbf{HK_H^+}$):

(NAME') $\vdash i \rightarrow \varphi \ / \ \vdash \varphi$, provided $i \notin \varphi$

(PASTE) $\vdash i : \Diamond j \wedge j : \varphi \rightarrow \psi \ / \ \vdash i : \Diamond \varphi \rightarrow \psi$, provided $i \neq j$ and $j \notin \varphi, \psi$

Proof of the (Bridge) and (MAME')

1. $\vdash i \wedge \neg\varphi \rightarrow i : \neg\varphi$ (Intro:)
2. $\vdash \Diamond(i \wedge \neg\varphi) \rightarrow \Diamond i : \neg\varphi$ (1 RM)
3. $\vdash \Diamond i \wedge \Box\neg\varphi \rightarrow \Diamond(i \wedge \neg\varphi)$ (K-thesis)
4. $\vdash \Diamond i \wedge \Box\neg\varphi \rightarrow \Diamond i : \neg\varphi$ (2,3 Syll)
5. $\vdash \Diamond i : \neg\varphi \rightarrow i : \neg\varphi$ (Back)
6. $\vdash \Diamond i \wedge \Box\neg\varphi \rightarrow i : \neg\varphi$ (4,5 Syll)
7. $\vdash \Diamond i \wedge \neg i : \neg\varphi \rightarrow \neg\Box\neg\varphi$ (6 CP)
8. $\vdash \Diamond i \wedge i : \varphi \rightarrow \Diamond\varphi$ (7 Selfdual:, Pos)

1. $\vdash i \rightarrow \varphi$ (Premise, $i \notin \varphi$)
2. $\vdash i : (i \rightarrow \varphi)$ (1 RG)
3. $\vdash i : i \rightarrow i : \varphi$ (2 K:)
4. $\vdash i : i$ (Ref:)
5. $\vdash i : \varphi$ (3,4 MP)
6. $\vdash \varphi$ (1,5 NAME)

Completeness

Theorem (Pure completeness): Let Γ be any set of pure-formulae, then $\mathbf{HK}^{+}_{\mathbf{H}} + \Gamma$ is strongly complete for the class of frames on which Γ is valid.

Definition:

- Γ is named iff it contains at least one nominal
- Γ is $\Diamond$-saturated iff for all $i : \Diamond\varphi \in \Gamma$, there is a nominal j such that $i : \Diamond j \in \Gamma$ and $j : \varphi \in \Gamma$

Lindenbaum Lemma Every $\mathbf{HK}^{+}_{\mathbf{H}}+\Gamma$-consistent set can be extended to a named, $\Diamond$-saturated, maximal, $\mathbf{HK}^{+}_{\mathbf{H}} + \Gamma$-consistent set.

Definition: Canonical Model for $\mathbf{HK}^{+}_{\mathbf{H}} + \Gamma$-maximal, consistent set Δ is defined as $\mathfrak{M}_c = \langle \mathcal{W}_c, \mathcal{R}_c, V_c \rangle$ where:

$\mathcal{W}_c = \{| i |: i$ is a nominal $\}$ and
$| i |= \{j : \; i : j \in \Delta\}$

$\mathcal{R}_c(| i |, | j |)$ iff $i : \Diamond j \in \Delta$

$V_c(p) = \{| i |: i : p \in \Delta\}$

$V_c(i) = \{| i |\}$

Truth Lemma $i : \varphi \in \Delta$ iff $\mathfrak{M}_c, | i | \models \varphi$

Frame Lemma if Δ is $\Diamond$-saturated $\mathbf{HK}^{+}_{\mathbf{H}} + \Gamma$-maximal, consistent set, then the frame of $\mathfrak{M}_c$ satisfies all properties defined by Γ

This result leads to better completeness theory due to more general theory of frame definability than standard modal logic provides. The following table lists some examples:

pure axioms		
name	axiom	frame-condition
(D')	$\Box i \to \Diamond i$	seriality
(DC')	$\Diamond i \to \Box i$	p. functionality
(T')	$\Box i \to i$	reflexivity
($\Box$T')	$\Box(\Box i \to i)$	almost-reflexivity
(IRR)	$i \to \Box \neg i$	irreflexivity
(4')	$\Box i \to \Box\Box i$	transitivity
(4C')	$\Box\Box i \to \Box i$	density
(INTR)	$\neg\Box i \to \Box\Box i$	intransitivity
(B')	$i \to \Box\Diamond i$	symmetry
(AS)	$i \to \Box\Box\neg i$	asymmetry
(ANT)	$i \to \Box(\Diamond i \to i)$	antisymmetry
(5')	$\Diamond i \to \Box\Diamond i$	euklideaness
(Un)	$\Diamond i$	universality
(3')	$\Box(\Box i \to j)$	
	$\vee\Box(\Box j \to i)$	s. connectedness
(L')	$\Box(\Box i \wedge i \to j)$	
	$\vee\Box(\Box j \wedge j \to i)$	w. connectedness
(DI)	$i : \Diamond j \vee j : \Diamond i$	dichotomy
(TRI)	$i : \Diamond j \vee j : \Diamond i \vee i : j$	trichotomy

Note in particular that:

1. Many conditions from the table are not definable in $\mathbf{L_M}$ e.g.: irreflexivity, intransitivity, asymmetry, antisymmetry, universality, dichotomy and trichotomy.

2. All conditions except (DI) and (TRI) are definable in $\mathbf{L_{H^*}}$.

3. Pure-formulae define only 1-order properties – but not all! for instance not all Sahlqvist formulae have pure formulae equivalents e.g. Church-Rosser property, Predecessors, Right-(Left)-directedness.

- Church-Rosser property (or connectedness):

  $\forall xyz(\mathcal{R}xy \wedge \mathcal{R}xz \rightarrow \exists v(\mathcal{R}yv \wedge \mathcal{R}zv))$

  is defined in $\mathbf{L_M}$ by (CR) $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$ but $\Diamond\Box i \rightarrow \Box\Diamond i$ doesn't work.

- Predecessors – $\forall x \exists y \mathcal{R} yx$ is not defined in $\mathbf{L_M}$ either (although the converse is).

- Right-directedness – $\forall xy \exists z(\mathcal{R}xz \wedge \mathcal{R}yz)$ is not definable in $\mathbf{L_M}$. It is definable in $\mathbf{L_H}$ by $i : \Box p \rightarrow j : \Diamond p$ but it is not pure-formula so pure-completeness theorem does not apply. Left-directedness is undefinable in $\mathbf{L_H}$ too.

Note! (CR) is a special case of a Geach Axiom: $\Diamond^m \Box^n \varphi \rightarrow \Box^s \Diamond^t \varphi$ which defines frame properties expressed in short by:

$$\forall xyz \exists v(\mathcal{R}^m xy \wedge \mathcal{R}^s xz \rightarrow \mathcal{R}^n yv \wedge \mathcal{R}^t zv)$$

Every instance of Geach Axiom is also an instance of Sahlqvist formula.

As a result we have a strange consequence:

Theorem (Pure completeness): Let Γ be any set of pure-formulae, then $\mathbf{HK}^{+}_{\mathbf{H}} + \Gamma$ is strongly complete for the class of frames defined by Γ.

Theorem (Sahlqvist completeness): Let Γ be any set of Sahlqvist-formulae, then $\mathbf{HK}^{+}_{\mathbf{H}}+\Gamma$ is strongly complete for the class of frames defined by Γ.

but

Completeness fails for some combinations of pure- and Sahlqvist-formulae! e.g. $\mathbf{HK}^{+}_{\mathbf{H}}+(CR)+(NG)$ is incomplete

where (NG) $\Diamond(i \wedge \Diamond j) \rightarrow \Box(\Diamond j \rightarrow i)$ defines

$\forall xyzu(\mathcal{R}xy \wedge \mathcal{R}xz \wedge \mathcal{R}yu \wedge \mathcal{R}zu \rightarrow y = z)$

The impact of past operators

$\mathbf{L_{TH}}$ is strictly more expressive than $\mathbf{L_H}$:

1. : is dispensable in the presence of past-operators, e.g. trichotomy may be defined by $Pi \vee i \vee Fi$.

2. Some frame-conditions undefinable in $\mathbf{L_H}$ by pure-formulae (although definable in $\mathbf{L_M}$) are definable, e.g. Church-Rosser property (or connectedness) is defined by $Fi \wedge Fj \rightarrow F(i \wedge FPj)$.

3. Some frame-conditions are definable that are not definable in any of $\mathbf{L_M}$, $\mathbf{L_T}$, $\mathbf{L_H}$, e.g.:

– left directedness $\forall xy \exists z(z < x \wedge z < y)$ is defined by PFi

– right discreteness

$$\forall xy(x < y \to \exists z(x < z \wedge \neg\exists v(x < v < z)))$$

is defined by $i : (F\top \to FHH\neg i)$
(or $i \to (F\top \to FHH\neg i)$)

In fact every Sahlqvist-formula have pure-formula equivalent in $\mathbf{L_{TH}}$, so we have:

Theorem (Sahlqvist/pure completeness): Let Γ be any set of Sahlqvist- or pure-formulae, then $\mathbf{HK_H^+}+\Gamma$ is strongly complete for the class of frames defined by Γ.

Disadvantages – $\mathbf{Kt_H}$ is still decidable but EXPTIME-complete, whereas $\mathbf{Kt}$ is in PSPACE (as $\mathbf{K}$ and $\mathbf{K_H}$)

LANGUAGE EXTENSIONS

- Extra modalities
  1. Global modalities
  2. Difference modalities
- Modal Binders
- 1-order modal hybrid logic

Global Modality

$\mathbf{L_{HA}}$ is $\mathbf{L_H}$ with universal modality A or (interdefinable) existential modality E defined as follows:

$\mathfrak{M}, w \models A\varphi$ iff $\mathfrak{M}, w' \models \varphi$ for any w'

$\mathfrak{M}, w \models E\varphi$ iff $\mathfrak{M}, w' \models \varphi$ for some w'

Note that $\mathbf{L_{H*A}} = \mathbf{L_{HA}}$ since : is definable:

$$i : \varphi := A(i \to \varphi) := E(i \wedge \varphi)$$

$\mathbf{K_{HA}}$ is decidable but these modalities are very strong – even $\mathbf{K_A}$ (no nominals) is EXPTIME-complete. Both $\mathbf{K_{HA}}$ and $\mathbf{Kt_{HA}}$ are in the same complexity class as plain $\mathbf{K_A}$.

Difference Modality

$\mathbf{L_{MD}}$ is $\mathbf{L_M}$ with difference possibility D or (interdefinable) difference necessity $\bar{D}$ defined as follows:

$\mathfrak{M}, w \models D\varphi$ iff $\mathfrak{M}, w' \models \varphi$ for some $w' \neq w$

$\mathfrak{M}, w \models \bar{D}\varphi$ iff $\mathfrak{M}, w' \models \varphi$ for any $w' \neq w$

Note that $\mathbf{L_{MD}}$ is strictly stronger than $\mathbf{L_{MA}}$ since A is definable by $\bar{D}$ but not otherwise:

$A\varphi := \varphi \wedge \bar{D}\varphi$

In fact in $\mathbf{L_{MD}}$ we can even simulate nominals:

p is true at exactly one point iff
$Ep \wedge A(p \rightarrow \neg Dp)$ holds.

On the other hand D is eliminable in $\mathbf{L_{HAD}}$. As a result we have the following hierarchy of expressivity:

$$\mathbf{L_{MA}} < \mathbf{L_{MD}} = \mathbf{L_{MAD}} = \mathbf{L_{HAD}} = \mathbf{L_{HA}} = \mathbf{L_{H^*A}}$$

We have also:

$$\mathbf{K_{MD}} = \mathbf{K_{HA}}$$

Note also that $D\varphi$ is definable in $\mathbf{Kt4.3}$ by $P\varphi \vee F\varphi$, so on linear frames $\mathbf{L_{TH}}, \mathbf{L_{THD}}$ and $\mathbf{L_{T}}$ have the same expressivity.

Modal Binders

Why not to quantify over states?

So the next step is:

- add the third sort of atoms $SVAR = \{u, v, ...\}$ (state variables) to the basic hybrid language

- add some binders – quantifiers $\forall, \exists$ or local binder $\downarrow$

The definition of the frame and model is the same as for $\mathbf{L_H}$ but we need also the concept of assignment a for $\mathcal{M}$ which is a mapping $a : SVAR \longrightarrow W$. The satisfaction of the formula is now defined for a model and an assignment. In particular, for the new elements we have the following conditions:

$\mathfrak{M}, a, w \models u$ iff $w = a(u)$

$\mathfrak{M}, a, w \models \forall u\varphi$ iff $\mathfrak{M}, a^u_{w'}, w \models \varphi$ for all w'

$\mathfrak{M}, a, w \models \exists u\varphi$ iff $\mathfrak{M}, a^u_{w'}, w \models \varphi$ for some w'

$\mathfrak{M}, a, w \models \downarrow u\varphi$ iff $\mathfrak{M}, a^u_w, w \models \varphi$

where a^u_w is an *u-variant of* a, namely $a^u_w(u) = w$ and $a^u_w(v) = a(v)$ for any $v \neq u$.

Note that $\downarrow$ is self-dual

We should also admit free state-variables as arguments of :, so the more general condition is:

$\mathfrak{M}, a, w \models s : \varphi$ iff $\mathfrak{M}, a, w' \models \varphi$ where
$s \in NOM \cup SVAR$ and $\{w'\} = V(s)$ or $w' = a(s)$

Let $\mathbf{L_{H\forall}}, \mathbf{L_{H\downarrow}}, \mathbf{L_{H\downarrow\forall}}$ denote Hybrid languages with added binders and $\mathbf{L_{H^*\forall}}, \mathbf{L_{H^*\downarrow}}, \mathbf{L_{H^*\downarrow\forall}}$ respective languages without satisfaction operators.

The difference between $\downarrow$ and $\forall$ is between local and global binding. $\mathbf{L_{H\downarrow}}$ enable to name current state ($\downarrow$ binds state variable to current state)

Fact: $\mathbf{L_{H^*\forall}}$ is strictly stronger than $\mathbf{L_{H^*\downarrow}}$, since:

1. $\downarrow$ is definable in $\mathbf{L_{H^*\forall}}$: $\quad \downarrow u\varphi := \exists u(u \wedge \varphi)$

but

2. $\mathbf{L_{H^*\downarrow}}$ is preserved under generated submodels whereas $\mathbf{L_{H^*\downarrow\forall}}$ is not.

(the same applies to $\mathbf{L_{H\forall}}, \mathbf{L_{H\downarrow}}$)

Corrolary: $\mathbf{L_{H^*\forall}} = \mathbf{L_{H^*\downarrow\forall}}$ and $\mathbf{L_{H\forall}} = \mathbf{L_{H\downarrow\forall}}$

Definability of satisfaction operators

Note! : is not definable in $\mathbf{L_{H^*\forall}}$

Fact: $\mathbf{L_{H\forall}} = \mathbf{L_{H^*\downarrow A}} = \mathbf{L_{H^*\forall A}} = \mathbf{L_{H\downarrow A}} = \mathbf{L_{H\forall A}}$

Recall that A defines sat-operator; moreover:

1. $\forall$ is defined in $\mathbf{L_{H^*\downarrow A}}$:

$\forall u\varphi :=\downarrow vA\downarrow uA(v \to \varphi)$, where $v \notin \varphi$

and

2. A is defined in $\mathbf{L_{H\forall}}$:

$A\varphi := \forall u, u : \varphi$

Both $\mathbf{K_{H\downarrow}}$ and $\mathbf{K_{H\forall}}$ is undecidable (in fact even $\mathbf{K_{H^*\forall}}$ is undecidable!)

Axiomatization

Let $\mathbf{K_{H\forall}}, \mathbf{K_{H\downarrow}}, \mathbf{K_{H\downarrow\forall}}$ denote basic Hybrid logics with added binders. $\mathbf{HK_{H\downarrow}}$ is obtained from $\mathbf{HK_H}$ by addition of:

(DA) i : $(\downarrow u\varphi \leftrightarrow \varphi[u/i])$

(S-D↓) $\downarrow u\varphi \leftrightarrow \neg \downarrow u\neg\varphi$

Pure-completeness holds for $\mathbf{HK^+_{H\downarrow}}$ exactly as for $\mathbf{HK^+_H}$. What's more we can axiomatize $\mathbf{HK^+_{H\downarrow}}$ without (BG) and (NOME) but using more standard rules (no side conditions). Just add to $\mathbf{HK_{H\downarrow}}$:

$(Name\downarrow) \vdash \downarrow u(u \rightarrow \varphi) \rightarrow \varphi$, provided $u \notin \varphi$

$(BG\downarrow) \vdash i$: $\Box \downarrow u\ i$: $\Diamond u$

$(RG\downarrow) \vdash \varphi\ /\ \vdash \downarrow u\varphi$

Pure-completeness of $\mathbf{HK}^{+}_{\mathbf{H}\downarrow}$ opens the question if we have something more. There are two points worth noticing:

1. $\mathbf{L_H}$ is more expressive (than $\mathbf{L_M}$) at the level of frames but even $\mathbf{L_{H^*\downarrow}}$ is more expressive at the level of models! For example we can distinguish between reflexive and nonreflexive states in a model ($\downarrow u \Diamond u$ and $\downarrow u \neg\Diamond u$)

2. Binary temporal operators U (Until) and S (Since) are definable in $\mathbf{L_{H\downarrow}}$ or $\mathbf{L_{TH^*\downarrow}}$:

$$U(\varphi, \psi) := \downarrow u \Diamond \downarrow v(\varphi \wedge u : \Box(\Diamond v \rightarrow \psi))$$

$$U(\varphi, \psi) := \downarrow uF(\varphi \wedge H(Pu \rightarrow \psi))$$

$\mathfrak{M}, t \vDash U(\varphi, \psi)$ iff $\mathfrak{M}, t' \vDash \varphi$ for some t' such that $\mathcal{R}tt'$ and $\mathfrak{M}, t'' \vDash \psi$ for every t'' such that $\mathcal{R}tt''$ and $\mathcal{R}t''t'$

Tenses

Standard Priorean $\mathbf{L_T}$ already have deictic nature but has strong limitation in expressing language tenses.

$\mathbf{L_{TH}}$ yields referential perspective which makes possible to express Reichenbachian analysis of tenses (see the table).

The addition of $\downarrow$ enrich further the referential possibilities of basic hybrid logic by operating with *storing* and *retrieving* system. Moreover, : is eliminable in $\mathbf{L_{TH\downarrow}}$ in all nominal-free sentences.

refer.	tense	example	formula
E-R-S	Pluperfect	I had seen	$P(i \wedge P\varphi)$
E,R-S	Past	I saw	$P(i \wedge \varphi)$
R-E-S	F-in-the-P	I'd see	$P(i \wedge F\varphi)$
R-S,E	F-in-the-P	I'd see	$P(i \wedge F\varphi)$
R-S-E	F-in-the-P	I'd see	$P(i \wedge F\varphi)$
E-S,R	Perfect	I've seen	$P\varphi$
S,R,E	Present	I see	φ
S,R-E	Prospective	I'm going to	$F\varphi$
S-E-R	Fut. perfect	I'll have	$F(i \wedge P\varphi)$
S,E-R	Fut. perfect	I'll have	$F(i \wedge P\varphi)$
E-S-R	Fut. perfect	I'll have	$F(i \wedge P\varphi)$
S-R,E	Future	I'll see	$P(i \wedge \varphi)$
S-R-E	F-in-the-F	abiturus ero	$F(i \wedge F\varphi)$

where:

S – the point of speech

E – the point of event

R – the point of reference

Axiomatization of $\mathbf{K}_{\mathbf{H}^*\forall}$

$\mathbf{HK}_{\mathbf{H}^*\forall}$ consists of axioms and rules of $\mathbf{HK}$ plus the following axioms:

$(Q1)$ $\forall u(\varphi \to \psi) \to (\varphi \to \forall u\psi)$,
where $u \notin VF(\varphi)$

$(Q2)$ $\forall u\varphi \to \varphi[u/s]$

$(Name)$ $\exists u, u$

(Nom) $\forall u(\Diamond^m(u \wedge \varphi) \to \Box^n(u \to \varphi)),\ m, n \in \omega$

$(Barcan)$ $\forall u\Box\varphi \to \Box\forall u\varphi$

and

(Gen) if $\vdash \varphi$, then $\vdash \forall u\varphi$

Expressiveness of $\mathbf{K_{H\forall}}$

All properties not expressible as pure-formulae in $\mathbf{L_H}$ (e.g. Geach axioms, Directedness) are expressible in $\mathbf{L_{H\forall}}$ as a PUENF-formula $\forall u_1, ..., u_m \exists v_1, ..., v_n \varphi$, where φ has no quantifiers, propositional variables, nominals. e.g.:

- Church-Rosser property – $\forall u_1 u_2 u_3 \exists v$ $(u_1 : \Diamond u_2 \wedge u_1 : \Diamond u_3 \to u_2 : \Diamond v \wedge u_3 : \Diamond v)$
- Predecessors – $\forall u \exists v, v : \Diamond u$
- Right-directedness – $\forall u_1 u_2 \exists v (u_1 : \Diamond v \wedge u_2 : \Diamond v)$

Theorem: Frame condition is defined by PUENF-formula iff it is UE-closure of strongly bounded 1-order formula.

Conjecture: every Sahlqvist-formula is expressible by PUENF-formula

Additional completenss result for $\mathbf{HK_H}$

Every PUENF-formula (PF) $\forall u_1, ..., u_m \exists v_1, ..., v_n \varphi$ corresponds to existential saturation rule (RPF) of the form:

If $\vdash \varphi[u_1/i_1, ..., u_m/i_m, v_1/j_1, ..., v_n/j_n] \rightarrow \psi$, then $\vdash \psi$ provided $j_1, ..., j_n$ are distinct, unequal to $i_1, ..., i_m$ and do not occur in ψ

For example for Church-Rosser we have:

If $\vdash (i_1 : \Diamond i_2 \wedge i_1 : \Diamond i_3 \rightarrow i_2 : \Diamond j \wedge i_3 : \Diamond j) \rightarrow \psi$, then $\vdash \psi$, provided $j \notin \psi$ and $j \neq i_1, i_2, i_3$

Lemma: If (PF) defines $\mathcal{F}$, then (RPF) is admissible in $\mathcal{F}$

Theorem (Extended Pure completeness): Let Γ be any set of pure-formulae and R any set of existential saturation rules, then $\mathbf{HK_H^+} + \Gamma + R$ is strongly complete for the class of frames defined by Γ and R.

Stronger binders

$\mathfrak{M}, a, w \models \Pi u\varphi$ iff $\mathfrak{M}, a^u_{w'}, w' \models \varphi$ for all w'

$\mathfrak{M}, a, w \models \Sigma u\varphi$ iff $\mathfrak{M}, a^u_{w'}, w' \models \varphi$ for some w'

$\mathfrak{M}, a, w \models \Downarrow u\varphi$ iff $\mathfrak{M}, a^u_{w}, w' \models \varphi$ for some w'

where a^u_w is an *u-variant of* a, namely $a^u_w(u) = w$ and $a^u_w(v) = a(v)$ for any $v \neq u$.

If we add to $\mathbf{L_{H^*}}$ any of these binders we have the following hierarchy:

$\mathbf{L_{H^*\downarrow}} < \mathbf{L_{H^*\exists}} < \mathbf{L_{H^*\Downarrow}}$ and $\mathbf{L_{H^*A}} < \mathbf{L_{H^*\Sigma}} < \mathbf{L_{H^*\Downarrow}}$ since:

$\exists u\varphi := \Downarrow v \Downarrow u(v \wedge \varphi)$ where $v \notin \varphi$

$E\varphi := \Sigma u\varphi$ where $u \notin \varphi$

$\Sigma u\varphi := \Downarrow v \Downarrow u(u \wedge \varphi)$ where $v \notin \varphi$

but $\Downarrow u\varphi := \downarrow uE\varphi$, so $\mathbf{L_{H^*\downarrow A}} = \mathbf{L_{H^*\Downarrow}}$

Relations with First-order Language

1. Standard Translation ST

$$
\begin{array}{lll}
ST_t(s) & = & t = s \\
ST_t(p) & = & P(t) \\
ST_t(\neg\varphi) & = & \neg ST_t(\varphi) \\
ST_t(\varphi \wedge \psi) & = & ST_t(\varphi) \wedge ST_t(\psi) \\
ST_t(\Diamond\varphi) & = & \exists x(R(tx) \wedge ST_x(\varphi)) \\
ST_t(s : \varphi) & = & ST_s(\varphi) \\
ST_t(E\varphi) & = & \exists x ST_x(\varphi) \\
ST_t(\downarrow u\varphi) & = & \exists x(x = t \wedge ST_t(\varphi)) \\
ST_t(\exists u\varphi) & = & \exists u ST_t(\varphi) \\
ST_t(\Sigma u\varphi) & = & \exists u ST_u(\varphi) \\
ST_t(\Downarrow u\varphi) & = & \exists x\exists y(y = t \wedge ST_x(\varphi))
\end{array}
$$

where x, y are variables distinct from term t and not occuring in φ.

2. Hybrid Translation HT

$$
\begin{array}{lcl}
HT(R(tt')) & = & t : \Diamond t' \\
HT(P(t) & = & t : p \\
HT(t = t') & = & t : t' \\
HT(\neg\varphi) & = & \neg HT(\varphi) \\
HT(\varphi \wedge \psi) & = & HT(\varphi) \wedge HT(\psi) \\
HT(\exists u \varphi) & = & E \downarrow u HT(\varphi)
\end{array}
$$

$\mathbf{L_{H\downarrow}}$ is equivalent to strongly bounded fragment of 1-order language.

$\mathbf{L_{H\forall}}$ has full 1-order language expressivity.

Note: Strongly bounded fragment of 1-order language covers all formulae built up from atoms with the help of boolean constants and bounded quantification (i.e. $\exists y(Rxy \wedge \varphi)$ and $\forall y(Rxy \rightarrow \varphi)$) – the fragment invariant under generated submodels.

1-order Hybrid logic QMHL

1. Vocabulary of $\mathbf{L}_{\mathbf{H}\downarrow}$ is enriched with:

- denumerable set of 1-order variables $FVAR = \{x, y, ...\}$

- denumerable set of rigid constants $CON = \{c_1, c_2, ...\}$

- denumerable set of nonrigid constants $FUN = \{f_1, f_2, ...\}$

- denumerable set of predicate symbols of n-arity $PRED = \{P_1, P_2, ...\}$

- 1-order (possibilistic) quantifiers and equality predicate: $\forall, \exists, =$

2. the set of terms contains $FVAR, CON$ and is closed with the rule:

- if $f \in FUN$ and $s \in NOM \cup SVAR$, then $s : f$ is a term

Note! sat-operator is used to form both formulae and terms.

3. Models are structures of the form $\mathfrak{M} = \langle\ \mathcal{W}, \mathcal{R}, D, V\rangle$, where D is a nonempty constant domain and V is defined as follows:

– $V(c) \in D$

– $V(i) \in \mathcal{W}$

– $V(P^n) \subseteq D^n \times \mathcal{W}$

– $V(f) \in D^{\mathcal{W}}$

An assignment $a = a_n \cup a_f$, where
$a_n : SVAR \longrightarrow \mathcal{W}$ and $a_f : FVAR \longrightarrow D$

The interpretation I of the term τ in a model and under an assignment is defined as follows:

$I(x) = a(x) \qquad I(c) = V(c)$

$I(s : f) = V(f)(w)$, where $w = a(s)$ if $s \in SVAR$ and $\{w\} = V(s)$ if $s \in NOM$

new clauses for satisfaction are:

$\mathfrak{M}, a, w \models P^n(\tau_1, ..., \tau_n)$ iff
$\langle I(\tau_1), ..., I(\tau_n), w\rangle \in V(P^n)$

$\mathfrak{M}, a, w \models \tau_1 = \tau_2$ iff $I(\tau_1) = I(\tau_2)$

$\mathfrak{M}, a, w \models \forall x\varphi$ iff $\mathfrak{M}, a^x_o, w \models \varphi$ for all $o \in D$

$\mathfrak{M}, a, w \models \exists x\varphi$ iff $\mathfrak{M}, a^x_o, w \models \varphi$ for some $o \in D$

An example: let c = Caroline and f = Miss America, then the sentence "Caroline is the present Miss America" is expressed by $\downarrow u(c = u : f)$. One can check that:

$$\models \downarrow u(c = u : f) \rightarrow \downarrow uG(c = u : f)$$

but

$$\not\models \downarrow u(c = u : f) \rightarrow G \downarrow u(c = u : f)$$

Decidability and complexity

3 possible effects of changing ordinary modal theories into hybrid theories:

1. The same complexity class e.g. $\mathbf{K_H}$

2. Worse behaviour e.g. $\mathbf{Kt_H}$

3. Better behaviour – logics of some frame classes.

Recall basic complexity hierarchy:

$P \leq NP \leq PSPACE \leq EXPTIME$

Some concrete results

1. Bad impact of past operators:

Even $\mathbf{Kt_{H^*}}$ with one nominal is EXPTIME-complete whereas $\mathbf{Kt}$ is PSPACE-complete but addition of : and A do not change the complexity.

2. Transitive frames:

Hybrid modal logics of transitive frames are in PSPACE even with A (recall that $\mathbf{K_{HA}}$ is EXPTIME-complete). But $\mathbf{Kt4_H}$ is still EXPTIME-complete.

3. Linear frames:

Hybrid logics of linear frames are NP-complete even with $\downarrow$! (note that even $\mathbf{K_{H^*\downarrow}}$ is undecidable).

Interpolation and Beth definability

L has the Strong Interpolation property iff: $\models_L \varphi \to \psi$ implies that $\models_L \varphi \to \chi$ and $\models_L \chi \to \psi$ for some χ such that $P(\chi) \subseteq P(\varphi) \cap P(\psi)$.

L has the Weak Interpolation property iff: $\varphi \mid\!\models \psi$ implies that $\varphi \mid\!\models_L \chi$ and $\chi \mid\!\models_L \psi$ for some χ such that $P(\chi) \subseteq P(\varphi) \cap P(\psi)$.

where P is the set of propositional variables and nominals.

Theorem: If $\models$ is compact, then strong interpolation implies weak interpolation.

Theorem: $\mathbf{K_{H\downarrow}}$ has strong interpolation; $\mathbf{K_H}$ has only weak interpolation.

example: there is no interpolant for $i \wedge \Diamond i \to (j \to \Diamond j)$ but if we limit P to propositional variables only, then strong interpolation holds also for $\mathbf{K_H}$.

There is an interesting relation between decidability and interpolation:

Theorem: Every hybrid logic built in extension of $\mathbf{L}_{\mathbf{H}^*}$ either is decidable or has strong interpolation (over nominals).

Theorem: $\mathbf{K}_{\mathbf{H}\downarrow}$ is the least logic with strong interpolation; any extension axiomatisable by a set of nominal-free sentences also has this property.

Good behaviour of $\mathbf{QMHL}$ in $\mathbf{L}_{\mathbf{H}\downarrow}$:

Theorem: Strong interpolation (and Beth definability) holds for any $\mathbf{QMHL}$ between $\mathbf{K}$ and $\mathbf{S5}$.

This result is in strong contrast to ordinary $\mathbf{QML}$ where we have:

Theorem (Fine): Interpolation fails for any $\mathbf{QML}$ between $\mathbf{K}$ and $\mathbf{S5}$ with constant domains and for $\mathbf{S5}$ with varying domains.

PROOF METHODS

A. General application of labels (some examples):

- the set of assumptions for a formula (e.g. Anderson/Belnap ND-systems for relevant logics)

- the set of truth values for a formula (e.g. Hahnle tableau systems for many-valued logics)

- possible world (point of time) satisfying a formula in modal (temporal) logics

B. Three kinds of labelled deduction

1. External – labels as an additional technical apparatus

2. Internalized – labels as a part of a language (in particular nominals in hybrid languages)

3. Mixed – both nominals (in a language) and labels (metalinguistic devices) present:

In External approach a variety of solutions:

1. Weak labelling – labels as a very limited device supporting proof construction, e.g. 3-labels tableau systems of Marx, Mikulas, Reynolds for Linear Tense Logics, Multisequent Calculi of Indrzejczak for Temporal Logics.

2. Strong labelling – system of labels as an exact representation of an attempted falsifying model, fusion of 2 systems: object language calculus + algebra of labels calculus, e.g. Gabbay's theory of labelled systems, Basin, Mathews, Vigano ND-systems for nonclassical logics.

3. Medium labelling – no special calculus for labels but still sufficient for construction of falsifying model e.g. Fitting's tableau calculi for modal logics.

Internalized and mixed systems for Hybrid Logics:

1. Ordinary calculi: Seligman's sequent calculus, ND-system of Indrzejczak.

2. Sat-calculi (rules defined on sat-formulae): Blackburn's Tableau system, Brauner's ND-system, Areces' Hylores-resolution system, HRND-system (Hybrid resolution-ND) of Indrzejczak.

3. Mixed calculi (with external labels): sequent system of Seligman and tableau system of Tzakova.

SELIGMAN'S SEQUENT CALCULUS FOR $\mathbf{K_{H\downarrow}}$

general rules

(AX) $\Gamma \Rightarrow \Delta$, where $\Gamma \cap \Delta \neq \varnothing$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\neg)\ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$(\wedge\Rightarrow)\ \frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\wedge)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\wedge\psi}$$

$$(\vee\Rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Gamma \Rightarrow \Delta}{\varphi\vee\psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\vee)\ \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi\vee\psi}$$

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi\rightarrow\psi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\rightarrow\psi}$$

nominal rules

$(N_1)^1 \quad \dfrac{i, j, \Gamma[i] \Rightarrow \Delta[i]}{i, j, \Gamma[j] \Rightarrow \Delta[j]} \qquad (N_2)^2 \quad \dfrac{\Gamma \Rightarrow \Delta}{i, \Gamma \Rightarrow \Delta}$

$(N_3)^3 \quad \dfrac{i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad (N_4)^2 \quad \dfrac{i, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$

Side conditions:

1. where $\Gamma[i]$ means that i occur in Γ and $\Gamma[j]$ is the result of replacement of j for i in Γ

2. where all elements of $\Gamma \cup \Delta$ are sat-formulae

3. where i does not occur in $\Gamma \cup \Delta$.

modal rules

$$(:I\Rightarrow)\ \frac{i, \varphi, \Gamma \Rightarrow \Delta}{i,\ i:\varphi,\ \Gamma \Rightarrow \Delta} \qquad (\Rightarrow:I)\ \frac{i, \Gamma \Rightarrow \Delta, \varphi}{i, \Gamma \Rightarrow \Delta,\ i:\varphi}$$

$$(:E\Rightarrow)\ \frac{i,\ i:\varphi, \Gamma \Rightarrow \Delta}{i, \varphi,\ \Gamma \Rightarrow \Delta} \qquad (\Rightarrow:E)\ \frac{i, \Gamma \Rightarrow \Delta,\ i:\varphi}{i, \Gamma \Rightarrow \Delta, \varphi}$$

$$(\Diamond\Rightarrow)^1\ \frac{\Diamond i,\ i:\varphi, \Gamma \Rightarrow \Delta}{\Diamond\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Diamond)\ \frac{\Gamma \Rightarrow \Delta, i:\varphi \quad \Gamma \Rightarrow \Delta, \Diamond i}{\Gamma \Rightarrow \Delta, \Diamond\varphi}$$

$$(\Box\Rightarrow)\ \frac{i:\varphi, \Gamma \Rightarrow \Delta \quad \Diamond i, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\Box)^1\ \frac{\Diamond i, \Gamma \Rightarrow \Delta, i:\varphi}{\Gamma \Rightarrow \Delta, \Box\varphi}$$

$$(\downarrow\Rightarrow)\ \frac{i, \varphi[x/i], \Gamma \Rightarrow \Delta}{i,\ \downarrow x\varphi,\ \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\downarrow)\ \frac{i, \Gamma \Rightarrow \Delta, \varphi[x/i]}{i, \Gamma \Rightarrow \Delta,\ \downarrow x\varphi}$$

Side condition:

1. where i does not occur in $\Gamma \cup \Delta \cup \{\varphi\}$.

Features of SC:

- No restriction on formulae in sequents

- cut is admissible

$$(Cut) \quad \frac{\Gamma \Rightarrow \Delta, \varphi \qquad \varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

ORDINARY NATURAL DEDUCTION FOR $\mathbf{K_H}$

1. Standard ND-system for **CPL**

Inference rules

(αE) α / α_i, where $i \in \{1,2\}$

(αI) α_1 , α_2 / α

(βE) β , $-\beta_i$ / β_j , where $i \neq j \in \{1,2\}$

(βI) β_i / β , where $i \in \{1,2\}$

α	α_1	α_2	β	β_1	β_2
$\varphi \wedge \psi$	φ	ψ	$\neg(\varphi \wedge \psi)$	$\neg\varphi$	$\neg\psi$
$\neg(\varphi \vee \psi)$	$\neg\varphi$	$\neg\psi$	$\varphi \vee \psi$	φ	ψ
$\neg(\varphi \to \psi)$	φ	$\neg\psi$	$\varphi \to \psi$	$\neg\varphi$	ψ

(⊥) φ , $-\varphi$ / ⊥

(¬¬) $\neg\neg\varphi$ / φ

Proof Construction rules

[COND] If $\Gamma, -\beta_i \vdash \beta_j$, then $\Gamma \vdash \beta$

[RED] if $\Gamma,\ -\varphi \vdash \bot$, then $\Gamma \vdash \varphi$

	Γ			Γ	
i	SHØW: β		i	SHØW: φ	
$i+1$		$-\beta_i$	$i+1$		$-\varphi$
		.			.
		.			.
k		β_j	k		$\bot$

2. Fitch-style ND-rules for **K**

[NEC] If $\Gamma \vdash \nu$, then $\Box\Gamma \vdash \nu^i$

[POS] if $\Gamma,\ \pi_1 \vdash \pi_2$, then $\Box\Gamma,\ \pi_1^i \vdash \pi_2^i$

where $\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\} \cup \{\neg\Diamond\varphi : \neg\varphi \in \Gamma\}$

$\Box\Gamma$			$\Box\Gamma \cup \{\pi_1^i\}$
i	SHØW: ν^i	i	SHØW: π_2^i
	Γ	$i+1$	π_1
	.		Γ
	.		.
k	ν	k	π_2

π^i	ν^i	$\pi = \nu$
$\Diamond\varphi$	$\Box\varphi$	φ
$\neg\Box\varphi$	$\neg\Diamond\varphi$	$\neg\varphi$

3. Inference Hybrid rules:

$(S - D)$ $\neg i : \varphi \ // \ i : \neg\varphi$

$(\bot :)$ $i : \bot \ // \ \bot$

$(: I)$ $i, \ \varphi \ / \ i : \varphi$

$(: E)$ $i, \ i : \varphi \ / \ \varphi$

$(I : E)$ $j : i : \varphi \ // \ i : \varphi$

(Ref) $\varnothing \ / \ i : i$

$(\Diamond E)$ $\Diamond i : \varphi \ / \ i : \varphi$

$(\Box I)$ $i : \varphi \ / \ \Box i : \varphi$

$(\vdash : E)$ $i : \varphi \ / \ \varphi$, provided $\vdash i : \varphi$

4. Hybrid Proof construction rules

[:I] If $\Gamma \vdash \varphi$, then $i : \Gamma \vdash i : \varphi$, where $i : \Gamma = \{i : \varphi : \varphi \in \Gamma\}$

[: $\Box$] if $\Gamma,\ i : \Diamond j \vdash j : \varphi$, then $\Gamma \vdash i : \Box\varphi$, where j is not in φ or in any undischarged assumption in Γ

	$i : \Gamma$			Γ
i	SHØW: $i : \varphi$		i	SHØW: $i : \Box\varphi$
	Γ		$i+1$	$i : \Diamond j$
	.			Γ
	.			.
k	φ		k	$j : \varphi$

1	SHØW: $i : j \wedge j : p \rightarrow i : p$	[6, $COND$]
2	$i : j \wedge j : p$	$ass.$
3	$i : j$	(2, αE)
4	$j : p$	(2, αE)
5	$i : j : p$	(4, $I : E$)
6	SHØW: $i : p$	[9, $: I$]
7	$j : p$	(5, $Reit.$)
8	j	(3, $Reit.$)
9	p	(7, 8 $: E$)

1	SHØW: $i : \Box p \wedge i : \Diamond j \rightarrow j : p$	[13, $COND$]
2	$i : p \wedge i : \Diamond j$	$ass.$
3	$i : \Box p$	(2, αE)
4	$i : \Diamond j$	(2, αE)
5	SHØW: $i : j : p$	[12, $: I$]
6	$\Box p$	(3, $Reit.$)
7	$\Diamond j$	(4, $Reit.$)
8	SHØW: $\Diamond j : p$	[11, POS]
9	j	$Mod.ass.$
10	p	(6, $Reit.$)
11	$j : p$	(9, 10 $: I$)
12	$j : p$	(8, $\Diamond E$)
13	$j : p$	(5, $I : E$)

BRAUNER'S NATURAL DEDUCTION FOR $\mathbf{K_{H\downarrow}}$

1. Inference rules

$(\wedge E)\ i : (\varphi \wedge \psi)\ /\ i : \varphi,\ i : \psi$

$(\wedge I)\ i : \varphi,\ i : \psi\ /\ i : (\varphi \wedge \psi)$

$(\rightarrow E)\ i : (\varphi \rightarrow \psi),\ i : \varphi\ /\ i : \psi$

$(\bot I)\ i : \bot\ /\ j : \bot$

$(: I)\ i : \varphi\ /\ j : i : \varphi$

$(: E)\ j : i : \varphi\ /\ i : \varphi$

$(Ref)\ \ \varnothing\ /\ i : i$

$(Nom_1)\ i : j,\ i : \varphi\ /\ j : \varphi,$ where $\varphi \in AT$

$(Nom_2)\ i : j,\ i : \Diamond k\ /\ j : \Diamond k$

$(\Box E)\ i : \Box\varphi,\ i : \Diamond j\ /\ j : \varphi$

2. Proof construction rules

[Cond] If $\Gamma,\ i : \varphi \vdash i : \psi$, then $\Gamma \vdash i : \varphi \to \psi$

[RAA] If $\Gamma,\ i : \neg\varphi \vdash i : \bot$, then $\Gamma \vdash i : \varphi$, where $\varphi \in VAR$

[$\Box$] if $\Gamma,\ i : \Diamond j \vdash j : \varphi$, then $\Gamma \vdash i : \Box\varphi$, where j is not in φ or in any undischarged assumption in Γ

for $\downarrow$ we need:

$(\downarrow E)$ $i :\downarrow x\varphi,\ i : j\ /\ j : \varphi[x/j]$

[$\downarrow I$] if $\Gamma,\ i : j \vdash j : \varphi[x/j]$, then $\Gamma \vdash i :\downarrow x\varphi$, where j is not in φ or in any undischarged assumption in Γ

Remarks:

1. It is Sat-Calculus, since it is defined on sat-formulae only. Sufficient generality of such approach follows from from the admissibility of:

(NAME) $\vdash i : \varphi \ / \ \vdash \varphi$ if $i \notin \varphi$

So if we want to prove a thesis φ which is not sat-formula we must try to prove $i : \varphi$ with $i \notin \varphi$.

2. The calculus is $\neg$-free, with $\bot$ instead, we can add suitable rules.

3. Normalization theorem holds

TABLEAU SYSTEMS

($\neg$) $\neg i : \neg\varphi$ / $i : \varphi$ and $i : \neg\varphi$ / $\neg i : \varphi$

($\neg\bot$) $\neg i : \bot$ / $\top$

($\neg\top$) $\neg i : \top$ / $\bot$

(α) α / α_1, α_2, see α below

(β) β / $\beta_1 \parallel \beta_2$, see β below

α	α_1	α_2
$i : (\varphi \wedge \psi)$	$i : \varphi$	$i : \psi$
$\neg i : (\varphi \vee \psi)$	$\neg i : \varphi$	$\neg i : \psi$
$\neg i : (\varphi \rightarrow \psi)$	$i : \varphi$	$\neg i : \psi$
β	β_1	β_2
$\neg i : (\varphi \wedge \psi)$	$\neg i : \varphi$	$\neg i : \psi$
$i : (\varphi \vee \psi)$	$i : \varphi$	$i : \psi$
$i : (\varphi \rightarrow \psi)$	$\neg i : \varphi$	$i : \psi$

$(: E)$ $j : i : \varphi$ / $i : \varphi$

$(\neg : E)$ $\neg j : i : \varphi$ / $\neg i : \varphi$

(Ref) $\varnothing$ / $i : i$ provided i is on the branch

(Nom) $i : j,\ i : \varphi$ / $j : \varphi$

$(Bridge)$ $i : j,\ k : \Diamond i$ / $k : \Diamond j$

$(\Box E)$ $i : \Box\varphi,\ i : \Diamond j$ / $j : \varphi$

$(\neg\Box E)$ $\neg i : \Box\varphi$ / $i : \Diamond j,\ \neg j : \varphi$, j new on the branch

$(\Diamond E)$ $i : \Diamond\varphi$ / $i : \Diamond j,\ j : \varphi$, j new on the branch

$(\neg\Diamond E)$ $\neg i : \Diamond\varphi,\ i : \Diamond j$ / $\neg j : \varphi$

$(\downarrow E)$ $i :\downarrow x\varphi$ / $i : \varphi[x/i]$

$(\neg\downarrow E)$ $\neg i :\downarrow x\varphi$ / $\neg i : \varphi[x/i]$

HYLORES

(Res) $\Gamma, i : \varphi$; $\Delta, i : \neg\varphi$ / Γ, Δ

($\wedge$) $\Gamma,\ i : \varphi \wedge \psi$ / $\Gamma,\ i : \varphi$; $\Gamma,\ i : \psi$

($\vee$) $\Gamma,\ i : \varphi \vee \psi$ / $\Gamma,\ i : \varphi,\ i : \psi$

($\Diamond$) $\Gamma, i : \Diamond\varphi$ / $\Gamma, i : \Diamond j$; $\Gamma, j : \varphi$, where j is a new nominal and $\varphi \notin NOM$

($\Box$) $\Gamma, i : \Diamond j$; $\Delta, i : \Box\varphi$ / $\Gamma, \Delta, j : \varphi$

(:) $\Gamma, i : j : \varphi$ / $\Gamma, j : \varphi$

(Ref) $\Gamma, i : \neg i$ / Γ

(Sym) $\Gamma, i : j$ / $\Gamma, j : i$

(Param) $\Gamma, i : j$; $\Delta, \varphi(i)$ / $\Gamma, \Delta, \varphi(i/j)$

Note that:

1. It is Resolution sat-calculus i.e. defined on clauses containing only sat-formulae.

2. Clauses are in generalised form; they contain not only literals prefixed with i : but any sat-formulae. ";" is used to separate clauses (it works like metalinguistic $\wedge$) and "," is used to separate elements in clauses (works like $\vee$).

3. All the formulae are assumed in negation normal form, so rules for negation are dispensable.

4. conditions on selection etc. are omitted in the above presentation.

RND – RESOLUTION BASED ND

ND sat-calculus defined on generalised clauses consists of:

1. Sat-versions of classical inference rules

(W) Γ / $\Gamma, i : \varphi$

($\neg$) $\Gamma, \neg i : \varphi$ // $\Gamma, i : \neg\varphi$

(Rez) $\Gamma, i : \varphi$; $\Gamma, i : -\varphi$ / Γ

(NN) $\Gamma, i : \neg\neg\varphi$ // $\Gamma, i : \varphi$

(α) $\Gamma, i : \alpha$ // $\Gamma, i : \alpha_1$; $\Gamma, i : \alpha_2$

(β) $\Gamma, i : \beta$ // $\Gamma, i : \beta_1, i : \beta_2$

2. Modal inference rules

(π^i) $\Gamma, i : \pi^i$ / $\Gamma, i : \Diamond_i j$; $\Gamma, j : \pi$, where j is new nominal in derivation

(ν^i) $\Gamma, i : \nu^i$; $\Delta, i : \Diamond_i j$ / $\Gamma, \Delta, j : \nu$

(:) $\Gamma, i : j : \varphi$ / $\Gamma, j : \varphi$

(:¬) $\Gamma, i : \neg j : \varphi$ / $\Gamma, \neg j : \varphi$

(Ref) $\Gamma, i : \neg i$ / Γ

(Sym) $\Gamma, i : j$ / $\Gamma, j : i$

(Nom) $\Gamma, i : j$; $\Delta, j : \varphi$ / $\Gamma, \Delta, i : \varphi$

(Bridge) $\Gamma, i : j$; $\Delta, k : \Diamond_i i$ / $\Gamma, \Delta, k : \Diamond_i j$

3. One proof-construction rule:

[Sub] if $X; -\varphi_1; \ldots; -\varphi_i$ / Δ , then X / Γ , where: Γ is nonempty, $\Delta \subseteq \Gamma$, $\{\varphi_1, ..., \varphi_i\} \subseteq \Gamma$, $i \geq 0$.

note: every φ in the schema is a sat-formula and X is a set of generalised clauses built from sat-formulae only.

X

k SHØW: Γ

$k+1$	$-\varphi_1$
.	.
.	.
$k+i$	$-\varphi_i$
.	.
.	.
n	Δ

Universality, generality and simplicity of RND

1. [Sub] is sufficiently general to cover all ND proof construction rules. In particular [Cond] is admissible as the following schema shows:

	X	
k	SHØW: β	$[n+1, Sub]$
$k+1$	SHØW: β_i, β_j	$[n, Sub]$
$k+2$	$-\beta_i$	z
	$\vdots$	
n	β_j	$(k+2, \ldots)$
$n+1$	β	$(k+1, \beta D')$

2. RND can simulate and combine proof search procedures from Resolution and Tableau based systems (like KE). It is due to the fact that RND applies Cut in both directions.

3. RND allows for extremely short and simple proofs – in classical logic no need of subderivations.

Examples

1	SHØW: $p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r)$	$[8, Cond]$
2	$p \vee (q \wedge r)$	ass
3	$p, q \wedge r$	$(2, \beta)$
4	p, q	$(3, \alpha)$
5	p, r	$(3, \alpha)$
6	$p \vee q$	$(4, \beta)$
7	$p \vee r$	$(5, \beta)$
8	$(p \vee q) \wedge (p \vee r)$	$(6, 7, \alpha)$

1	SHØW: $\neg(p \rightarrow (q \rightarrow r)), p \rightarrow r, \quad p \wedge \neg q$	$[10, Sub]$
2	$p \rightarrow (q \rightarrow r)$	ass
3	$\neg(p \rightarrow r)$	ass
4	$\neg p, q \rightarrow r$	$(2, \beta)$
5	$\neg p, \neg q, r$	$(4, \beta)$
6	p	$(3, \alpha)$
7	$\neg r$	$(3, \alpha)$
8	$\neg q, r$	$(5, 6, Rez)$
9	$\neg q$	$(7, 8, Rez)$
10	$p \wedge \neg q$	$(6, 9, \alpha)$

Comments on cut application

1. Resolution is a special case of forwards-application of Cut.

$$(Res) \quad \frac{\Gamma, \varphi \qquad -\varphi, \Delta}{\Gamma, \Delta}$$

2. In tableau systems we can have backwards-application of Cut e.g. in Hintikka-style systems it has a form:

$$(B - Cut) \qquad \frac{\Gamma}{\Gamma, \varphi \mid \Gamma, -\varphi}$$

3. Some systems (Davis-Putnam procedure, KE) involve both forms of Cut but in a very limited special way. In RND we have both forms in full generality since we have resolution (F-Cut) and [Sub] can simulate (B-Cut).

Extensions

We can provide 3 forms of rules:

- with 1-parameter-formula φ

  (1R-A) Γ, φ / Γ

- with 2-parameter-formulae φ and ψ

  (2Exp-A) Γ, φ / Γ, ψ or

  (2R-A) Γ, φ ; $\Delta, -\psi$ / Γ, Δ

- with 3-parameter-formulae φ, ψ and χ

  (3Exp-A) Γ, φ / Γ, ψ, χ or

  (3RExp-A) Γ, φ ; $\Delta, -\psi$ / Γ, Δ, χ or

  (3R-A) Γ, φ ; $\Delta, -\psi$; $\Sigma, -\chi$ / Γ, Δ, Σ

Theorem: Rules of the type (2Exp-A), (2R-A) and (3RExp-A), (3Exp-A), (3R-A) are inter-derivable

axiom	φ	ψ	χ
(DC)	$i : \Diamond j$	$\neg i : \Diamond k$	$j : k$
(T)	$\neg i : \Diamond i$	$-$	$-$
(I)	$i : \Diamond i$	$-$	$-$
(4)	$i : \Diamond j$	$\neg j : \Diamond k$	$i : \Diamond k$
(5)	$i : \Diamond j$	$\neg i : \Diamond k$	$j : \Diamond k$
(B)	$i : \Diamond j$	$j : \Diamond i$	$-$
(As)	$i : \Diamond j$	$\neg j : \Diamond i$	$-$
(As')	$i : \Diamond j$	$\neg j : \Diamond i$	$i : j$
(L)	$\neg i : \Diamond j$	$j : \Diamond i$	$-$
(L')	$\neg i : \Diamond j$	$j : \Diamond i$	$i : j$
(3)	$i : \Diamond j$	$\neg i : \Diamond k$	$j : \Diamond k,\ k : \Diamond j$
(3')	$i : \Diamond j$	$\neg i : \Diamond k$	$j : \Diamond k,\ k : \Diamond j,\ j : k$

Note that for (3) and (3') we have χ_1, χ_2 and χ_1, χ_2, χ_3 respectively instead of single χ. For instance (3RExp-3') has a form Γ, φ ; $\Delta, -\psi$ / $\Gamma, \Delta, \chi_1, \chi_2, \chi_3$.

Example

1	SHØW: $1 : (2 \rightarrow \Box(\Diamond 2 \rightarrow 2))$	$[12, Sub]$
2	$\neg 1 : (2 \rightarrow \Box(\Diamond 2 \rightarrow 2))$	ass
3	$1 : 2$	$(2, \alpha)$
4	$1 : \neg\Box(\Diamond 2 \rightarrow 2)$	$(2, \alpha)$
5	$1 : \Diamond 3$	$(4, \pi^i)$
6	$3 : \neg(\Diamond 2 \rightarrow 2)$	$(4, \pi^i)$
7	$3 : \Diamond 2$	$(6, \alpha)$
8	$3 : \neg 2$	$(6, \alpha)$
9	$\neg 3 : 2$	$(8, NN)$
10	$2 : 1$	$(3, Sym)$
11	$2 : \Diamond 3$	$(10, 5, Nom)$
12	$\bot$	$(7, 11, 9, 3R - As')$

Nominal existence rules

(HRDN-4C) $\Gamma, i : \Diamond j$ / $\Gamma, i : \Diamond k$; $\Gamma, k : \Diamond j$, where k is a new nominal

(HRDN-CR) $\Gamma, i : \Diamond j$; $\Delta, i : \Diamond k$ / $\Gamma, \Delta, j : \Diamond l$; $\Gamma, \Delta, k : \Diamond l$, where l is a new nominal

1	SHØW: $1 : (\Diamond\Box 2 \to \Box\Diamond 2)$	$[13, Sub]$
2	$\neg 1 : (\Diamond\Box 2 \to \Box\Diamond 2)$	ass
3	$1 : \Diamond\Box 2$	$(2, \alpha)$
4	$1 : \neg\Box\Diamond 2$	$(2, \alpha)$
5	$1 : \Diamond 3$	$(3, \pi^i)$
6	$3 : \Box 2$	$(3, \pi^i)$
7	$1 : \Diamond 4$	$(4, \pi^i)$
8	$4 : \neg\Diamond 2$	$(4, \pi^i)$
9	$3 : \Diamond 5$	$(5, 7, HRDN-CR)$
10	$4 : \Diamond 5$	$(5, 7, HRDN-CR)$
11	$5 : 2$	$(6, 9, \nu^i)$
12	$5 : \neg 2$	$(8, 10, \nu^i)$
13	$\bot$	$(11, 12, Rez)$

REFERENCES

Areces, C., J. Heguiabehere, ‘HyLoRes: Direct Resolution for Hybrid Logics‘, dostępne na stronie http://www.illc.uva.nl/~carlos/hybrid

Basin, D., S. Mathews, and L. Vigano, ‘Labelled Propositional Modal Logics: Theory and Practice‘, *Journal of Logic and Computation* 7(6):685–717, 1997.

Blackburn, P., ‘Internalizing Labelled Deduction‘, *Journal of Logic and Computation*, 10(1):137–168, 2000.

Blackburn, P., ‘Representation, Reasoning, and Relational Structures: a Hybrid Logic Manifesto‘, *Logic Journal of the IGPL*, 8(3):339–365, 2000.

Blackburn, P., M. DeRijke, and Y. Venema, *Modal Logic*, Cambridge University Press, Cambridge 2001.

Braüner, T., ‘Natural Deduction for Hybrid Logic‘, *Journal of Logic and Computation* 14/3:329–353, 2004.

Demri, S. i R. Goré, ‘Cut-free Display Calculi for Nominal Tense Logics‘, w: N. Murray (red.), **Tableaux 99, LNAI 1617**, pp. 155–170, Springer, Berlin 1999.

Fitting, M., *Proof Methods for Modal and Intuitionistic Logics*, Reidel, Dordrecht 1983.

Gabbay, D., *LDS - Labelled Deductive Systems*, Clarendon Press, Oxford 1996.

Goré, R. ‘Tableau Methods for Modal and Temporal Logics‘, in: M. D’Agostino et al. (eds.),

Handbook of Tableau Methods, pp. 297–396, Kluwer Academic Publishers, Dordrecht 1999.

Indrzejczak, A., ‘Multiple Sequent Calculus for Tense Logics‘, Abstracts of AiML and ICTL 2000:93–104, Leipzig 2000.

Indrzejczak, A., ‘Hybrid system for (not only) Hybrid Logic‘, Abstracts of AiML 2002:137–156, Toulouse 2002.

Indrzejczak, .A., ‘Resolution based Natural Deduction‘, *Bulletin of the Section of Logic* 31(3):159–170, 2002.

Marx, M., S. Mikulas, i M. Reynolds, ‘The Mosaic Method for Temporal Logics‘, w: R. Dyckhoff (red.) **Automated Reasoning with Analytic Tableaux and Related Methods**, Proc. of International Conference TABLEAUX

2000, Saint Andrews, Scotland, LNAI 1847:324–340, Springer 2000.

Perzanowski, J., ‘Logiki modalne a filozofia‘, w: J. Perzanowski (ed.) **Jak filozofować**, str. 262–346, PWN, Warszawa 1989.

Prior, A. *Time and Modality*, Oxford University Press, Oxford 1957.

Prior, A. *Past, Present and Future*, Clarendon/Oxford University Press, Oxford 1967.

Prior, A. *Papers on Time and Tense*, Clarendon/Oxford University Press, Oxford 1968.

Rescher, N., and A. Urquhart, *Temporal Logic*, Springer-Verlag, New York 1971.

Seligman, J., ‘The Logic of Correct Description‘ in: M. de Rijke (ed.) **Advances in Intensional Logic**, pp. 107–135. Kluwer 1997, Tzakova, M., ‘Tableaux Calculi for Hybrid Logics‘, in: N. Murray (ed.) **Conference on Tableaux Calculi and Related Methods (TABLEAUX), Saratoga Springs**, Springer 2002

HYBRID LOGIC HOMEPAGE:

http://hylo.loria.fr