SEQUENT CALCULI FOR WEAK MODAL LOGICS

Andrzej Indrzejczak

Department of Logic
University of Łódź
Kopcińskiego 16/18
90–232 Łódź
Poland
e-mail: indrzej@filozof.uni.lodz.pl

Basic rules and axioms

RE if $\varphi \leftrightarrow \psi \in \mathbf{L}$, then $\Box\varphi \leftrightarrow \Box\psi \in \mathbf{L}$

RM if $\varphi \rightarrow \psi \in \mathbf{L}$, then $\Box\varphi \rightarrow \Box\psi \in \mathbf{L}$

RC if $\varphi \wedge \psi \rightarrow \chi \in \mathbf{L}$, then $\Box\varphi \wedge \Box\psi \rightarrow \Box\chi \in \mathbf{L}$

RR if $\wedge\Gamma \rightarrow \psi \in \mathbf{L}$, then $\wedge\Box\Gamma \rightarrow \Box\psi \in \mathbf{L}$, where $\Gamma \neq \varnothing$

RN if $\varphi \in \mathbf{L}$, then $\Box\varphi \in \mathbf{L}$

M $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

C $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

N $\Box\top$

Modal Logics

Every **L** containing **CPL** and closed wrt RE is congruent (classical)

Every **L** containing **CPL** and closed wrt RM is monotonic (or every congruent **L** containing M)

Every **L** containing **CPL** and closed wrt RR is regular (or every monotonic **L** containing C)

Every regular **L** closed wrt RN is normal

Hierarchy

Let **E** denote the weakest congruent logic, **M** — the weakest monotonic, **R** — the weakest regular, and **K** — the weakest normal logic.

The Big Five

- D $\Box\varphi \to \neg\Box\neg\varphi$
- T $\Box\varphi \to \varphi$
- 4 $\Box\varphi \to \Box\Box\varphi$
- B $\varphi \to \Box\neg\Box\neg\varphi$
- 5 $\neg\Box\varphi \to \Box\neg\Box\varphi$

The Dependence Theorem

$\mathbf{CPL} + T \vdash D$

$\mathbf{CPL} + T + 5 \vdash B$

$\mathbf{CPL} + D + 4 + B \vdash T$

$\mathbf{E} + T + 5 \vdash 4$

$\mathbf{E} + B + 4 + D \vdash 5 \quad \mathbf{M} + B + 4 \vdash 5$

$\mathbf{E} + B + 5 + T \vdash 4 \quad \mathbf{M} + B + 5 \vdash 4$

$\mathbf{E} + B + T \vdash N \qquad \mathbf{M} + B \vdash N$

There is 18 different E-logics, 15 M-logics, 16 EN-logics and 10 MN-logics axiomatized with D, T, 4, B, 5 over **E** and **M** with possibly RN added

SC-CPL – rules

$(AX)\quad \varphi \Rightarrow \varphi$

$(Cut)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$

$(W\Rightarrow)\quad \dfrac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ $\qquad (\Rightarrow W)\quad \dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$

$(C\Rightarrow)\quad \dfrac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}$ $\qquad (\Rightarrow C)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$

$(\neg\Rightarrow)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi}{\neg\varphi, \Gamma \Rightarrow \Delta}$ $\qquad (\Rightarrow\neg)\quad \dfrac{\varphi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$

$(\wedge\Rightarrow)\quad \dfrac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi\wedge\psi, \Gamma \Rightarrow \Delta}$ $\qquad (\Rightarrow\wedge)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \Pi \Rightarrow \Sigma, \psi}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, \varphi\wedge\psi}$

$(\vee\Rightarrow)\quad \dfrac{\varphi, \Gamma \Rightarrow \Delta \quad \psi, \Pi \Rightarrow \Sigma}{\varphi\vee\psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$ $\qquad (\Rightarrow\vee)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma \Rightarrow \Delta, \varphi\vee\psi}$

$(\rightarrow\Rightarrow)\quad \dfrac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Pi \Rightarrow \Sigma}{\varphi\rightarrow\psi, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$ $\qquad (\Rightarrow\rightarrow)\quad \dfrac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi\rightarrow\psi}$

General Rules

(E) $\dfrac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \varphi}{\Box\varphi \Rightarrow \Box\psi}$ (M) $\dfrac{\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi}$

(C) $\dfrac{\varphi,\ \psi \Rightarrow \chi \quad \chi \Rightarrow \varphi \quad \chi \Rightarrow \psi}{\Box\varphi,\ \Box\psi \Rightarrow \Box\chi}$

(N) $\dfrac{\Rightarrow \varphi}{\Rightarrow \Box\varphi}$ (T) $\dfrac{\varphi,\ \Gamma \Rightarrow \Delta}{\Box\varphi,\ \Gamma \Rightarrow \Delta}$

corresponding to:

RE $\varphi \leftrightarrow \psi\ /\ \Box\varphi \leftrightarrow \Box\psi$

RM $\varphi \rightarrow \psi\ /\ \Box\varphi \rightarrow \Box\psi$
(or to M $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$)

RC $\varphi, \psi \rightarrow \chi\ /\ \Box\varphi, \Box\psi \rightarrow \Box\chi$
(or to C $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$)

RN $\varphi\ /\ \Box\varphi$ (or to N $\Box\top$)

T $\Box\varphi \rightarrow \varphi$

Special Rules

(D-2) $\dfrac{\Rightarrow \varphi, \psi \qquad \varphi, \psi \Rightarrow}{\Box\varphi, \Box\psi \Rightarrow}$ (D) $\dfrac{\varphi, \psi \Rightarrow}{\Box\varphi, \Box\psi \Rightarrow}$

(4-2) $\dfrac{\Box\varphi \Rightarrow \psi \qquad \psi \Rightarrow \Box\varphi}{\Box\varphi \Rightarrow \Box\psi}$ (4) $\dfrac{\Box\varphi \Rightarrow \psi}{\Box\varphi \Rightarrow \Box\psi}$

(5-2) $\dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Rightarrow \Box\varphi, \Box\psi}$ (5) $\dfrac{\Rightarrow \Box\varphi, \psi}{\Rightarrow \Box\varphi, \Box\psi}$

(B-2) $\dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Rightarrow \varphi, \Box\psi}$ (B) $\dfrac{\Rightarrow \Box\varphi, \psi}{\Rightarrow \varphi, \Box\psi}$

corresponding to:

- D $\Box\varphi \rightarrow \neg\Box\neg\varphi$
- 4 $\Box\varphi \rightarrow \Box\Box\varphi$
- 5 $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
- B $\varphi \rightarrow \Box\neg\Box\neg\varphi$

Equivalence of the systems

The case of 5 and (5-2)

(5-2) implies 5

$$
\dfrac{
  \dfrac{
    \dfrac{
      (\Rightarrow\neg)\,\dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Rightarrow \Box\varphi, \neg\Box\varphi}
      \qquad
      \dfrac{\Box\varphi \Rightarrow \Box\varphi}{\Box\varphi, \neg\Box\varphi \Rightarrow}\,(\neg\Rightarrow)
    }{\Rightarrow \Box\varphi, \Box\neg\Box\varphi}\,(\text{5-2})
  }{\neg\Box\varphi \Rightarrow \Box\neg\Box\varphi}\,(\neg\Rightarrow)
}{\Rightarrow \neg\Box\varphi \rightarrow \Box\neg\Box\varphi}\,(\Rightarrow\rightarrow)
$$

(E) + 5 implies (5-2)

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\mathcal{D}}{\neg\Box\varphi \Rightarrow \Box\neg\Box\varphi}
      \qquad
      \dfrac{
        (\neg\Rightarrow)\,\dfrac{\Rightarrow \Box\varphi, \psi}{\neg\Box\varphi \Rightarrow \psi}
        \qquad
        \dfrac{\Box\varphi, \psi \Rightarrow}{\psi \Rightarrow \neg\Box\varphi}\,(\Rightarrow\neg)
      }{\Box\neg\Box\varphi \Rightarrow \Box\psi}\,(E)
    }{\neg\Box\varphi \Rightarrow \Box\psi}\,(Cut)
  }{\Rightarrow \Box\psi, \neg\neg\Box\varphi}\,(\Rightarrow\neg)
}{\Rightarrow \Box\psi, \Box\varphi}\ (\text{double line})
$$

Additional Rules

(D'-2) $\dfrac{\Rightarrow \varphi, \psi \qquad \varphi, \psi \Rightarrow}{\Rightarrow \Box\varphi, \Box\psi}$ (D') $\dfrac{\Rightarrow \varphi, \psi}{\Rightarrow \Box\varphi, \Box\psi}$

(4'-2) $\dfrac{\varphi \Rightarrow \Box\psi \qquad \Box\psi \Rightarrow \varphi}{\Box\varphi \Rightarrow \Box\psi}$ (4') $\dfrac{\varphi \Rightarrow \Box\psi}{\Box\varphi \Rightarrow \Box\psi}$

(5'-2) $\dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\Box\varphi, \Box\psi \Rightarrow}$ (5') $\dfrac{\Box\varphi, \psi \Rightarrow}{\Box\varphi, \Box\psi \Rightarrow}$

(B'-2) $\dfrac{\Rightarrow \Box\varphi, \psi \qquad \Box\varphi, \psi \Rightarrow}{\varphi, \Box\psi \Rightarrow}$ (B') $\dfrac{\Box\varphi, \psi \Rightarrow}{\varphi, \Box\psi \Rightarrow}$

corresponding to:

- D' $\neg\Box\neg\varphi \rightarrow \Box\varphi$
- 4' $\Box\Box\varphi \rightarrow \Box\varphi$
- 5' $\Box\neg\Box\varphi \rightarrow \neg\Box\varphi$
- B' $\Box\neg\Box\neg\varphi \rightarrow \varphi$

Cut-elimination

In the class of M-logics cut is admissible in: **SC-M, SC-MD, SC-MT, SC-M4, SC-M5, SC-MT4, SC'-MD4, SC-M45, SC'-MD45**; the same holds for respective MN-logics

In the class of E-logics cut is admissible in: **SC-E, SC-ED, SC-ET, SC-E5, SC-ET4**; the same holds for EN-logics except **SC-END**

1. Both **E5** and **M5** (and their N-counterparts) have cut-free formalization in contrast to **R5** and **K5**

2. Neither **ED5** nor **MD5** (and their N-counterparts) have cut-free formalisation

3. Cut-elimination fails for all B-logics

4. In E-logics cut-elimination fails for almost all 4-logics (the exception is **ET4** in contrast to M-logics

Decision procedures for M-logics

$$\Gamma,\ \Box\varphi_1, ..., \Box\varphi_l \Rightarrow \Box\psi_1, ..., \Box\psi_k,\ \ \Delta,$$

where $\Gamma \cup \Delta \subseteq VAR$, $\Gamma \cap \Delta = \emptyset$ and $l + k > 0$

Characteristic sets of sequents:

$\mathcal{S}_M = \{\varphi_i \Rightarrow \psi_j\ :\ i \leq l,\ j \leq k\}$

$\mathcal{S}_4 = \{\Box\varphi_i \Rightarrow \psi_j\ :\ i \leq l,\ j \leq k\}$

$\mathcal{S}_5 = \{\Rightarrow \psi_i,\ \psi_j\ :\ i \leq k,\ j \leq k\}$

$\mathcal{S}_D = \{\varphi_i,\ \varphi_j \Rightarrow\ :\ i \leq l,\ j \leq l\}$

$\mathcal{S}_{D5} = \{\varphi_i \Rightarrow \Box\psi_j\ :\ i \leq l,\ j \leq k\}$

$\mathcal{S}_{D4} = \{\varphi_i,\ \Box\varphi_j \Rightarrow\ :\ i \leq l,\ j \leq l\}$

$\mathcal{S}_L$ - The set of subproof generators for each logic having cut-free formalization:

Calculus	$\mathcal{S}_L$
SC-M	$\mathcal{S}_M$
SC-MT	$\mathcal{S}_M$
SC-MD	$\mathcal{S}_M \cup \mathcal{S}_D$
SC-M4	$\mathcal{S}_M \cup \mathcal{S}_4$
SC-MT4	$\mathcal{S}_M \cup \mathcal{S}_4$
SC-M5	$\mathcal{S}_M \cup \mathcal{S}_5$
SC*-MD4	$\mathcal{S}_M \cup \mathcal{S}_D \cup \mathcal{S}_4 \cup \mathcal{S}_{D4}$
SC-M45	$\mathcal{S}_M \cup \mathcal{S}_4 \cup \mathcal{S}_5$
SC*-MD45	$\mathcal{S}_M \cup \mathcal{S}_D \cup \mathcal{S}_4 \cup \mathcal{S}_5 \cup \mathcal{S}_{D4} \cup \mathcal{S}_{D5}$

The meta-rule of Subtree Generation (SG)

$$\text{(SG)} \quad \frac{\mathcal{S}_L}{\Gamma,\ \Box\varphi_1, ..., \Box\varphi_l \Rightarrow \Box\psi_1, ..., \Box\psi_k,\ \Delta}$$

Note! In **MD4** and **MD45** possible loops due to repetition of sequents of the form $\varphi, \Box\varphi \Rightarrow$ (cf. the definition of $\mathcal{S}_{D4}$) but in **M4, MT4, M45** no risk of infinite branches!